

QCD-scale modified-gravity universe

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Abstract

A possible gluon-condensate-induced modified-gravity model with $f(R) \propto |R|^{1/2}$ has been suggested previously. Here, a simplified version is presented using the constant flat-spacetime equilibrium value of the QCD gluon condensate and a single pressureless matter component (cold dark matter, CDM). The resulting dynamical equations of a spatially-flat and homogeneous Robertson–Walker universe are solved numerically. This simple empirical model allows, in fact, for a careful treatment of the boundary conditions and does not require a further scaling analysis as the original model did. Reliable predictions are obtained for several observable quantities of the homogeneous model universe. In addition, the estimator E_G , proposed by Zhang et al. to search for deviations from standard Einstein gravity, is calculated for linear sub-horizon matter-density perturbations. The QCD-scale modified-gravity prediction for $E_G(z)$ differs from that of the Λ CDM model by about $\pm 10\%$ depending on the redshift z .

Key words: modified theories of gravity, cosmology, dark energy

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1. Introduction

It has been suggested that the gluon condensate (Shifman, Vainshtein & Zakharov, 1979; Narison, 1996) of quantum chromodynamics [QCD] in the q -theory framework (Klinkhamer & Volovik, 2008a,b, 2010) may lead to a particular type of $f(R)$ modified-gravity model (Klinkhamer & Volovik, 2009a). The cosmology of this model was explored in Klinkhamer (2010a). Here, the model is simplified even further, allowing for a detailed study of the corresponding cosmological equations and eliminating all arbitrariness.

The model of this article is, however, considered from a different point of view than the one used in the original article (Klinkhamer & Volovik, 2009a). Here, the aim is not to derive the particular model used from the underlying theory (general relativity and QCD), but to consider the model *per se* and to test the hypothesis whether or not this model gives a reasonably accurate description of the observed “accelerating universe” (cf. Riess et al., 1998; Perlmutter et al., 1999; Komatsu et al., 2009, and references therein). In other words, the approach is in the spirit of Kepler rather than Newton.

Still, the model is not chosen at random from all possible $f(R)$ modified-gravity models, but obeys certain physically motivated selection criteria, as discussed in Section 2.1. The selected QCD-scale modified-gravity model is detailed in Section 2.2, with a few technical points relegated to Section 2.3.

The corresponding cosmological model with a single matter component (cold dark matter, CDM) is introduced in Section 3.1. The resulting differential equations are given in Section 3.2. Three observable quantities of the homogeneous cosmological model are then discussed in Section 3.3. A fourth observable, this time of the inhomogeneous cosmological model, is also considered, namely, the estimator E_G , which was designed (Zhang et al., 2007) to characterize the linear growth of matter-density perturbations and which has been measured recently for moderate redshifts (Reyes et al., 2010). In Section 4, the same four observable quantities are calculated analytically for the standard Λ CDM-model universe (Weinberg, 1972; Carroll, Press, & Turner, 1992; Sahni & Starobinsky, 2000; Peebles & Ratra, 2003; Perivolaropoulos, 2010) and may serve as benchmark results.

The numerical results for both homogeneous and inhomogeneous observable quantities of the QCD-scale modified-gravity universe are discussed in Section 5 and compared with those of the Λ CDM universe. Final comments are presented in Section 6.

Even though not necessary for a proper understanding of the present article, the reader may wish to have a look at Appendix A of a recent review (Klinkhamer & Volovik, 2011), where the basic idea of q -theory is outlined.

2. QCD-scale modified-gravity model

2.1. Selection criteria

There exists an infinity of $f(R)$ modified-gravity models and an infinity thereof reproduces the main characteristics of the observed homogeneous universe [see, e.g., Brans & Dicke (1961); Starobinsky (1980); Bertolami (1986); Sotiriou & Faraoni (2010)]. In this paper, a subset of these theories is obtained by imposing the following conditions:

1. Empty flat spacetime is a solution of the field equations resulting from the action with the function $f(R)$ replaced by the constant $f(0)$.
2. The action of the $f(R)$ modified-gravity model involves only known energy scales from general relativity and the standard model of elementary particle physics [e.g., the energy scales $E_{\text{QCD}} = \text{O}(10^2 \text{ MeV})$ and $E_{\text{Planck}} \equiv \sqrt{\hbar c^5/(8\pi G_N)} \approx 2.44 \times 10^{18} \text{ GeV}$, for the particular model of the next subsection].
3. The asymptotic de-Sitter solution from the $f(R)$ modified-gravity model has only integer powers of the Hubble constant H in the reduced action [e.g., the terms $(E_{\text{Planck}})^2 H^2$ and $(E_{\text{QCD}})^3 H$, for the particular model of the next subsection].
4. The action of the $f(R)$ modified-gravity model has dimensionless coupling constants roughly of order unity [e.g., the coupling constant η , for the particular model of the next subsection].

The first condition is motivated by the need to solve the main cosmological constant problem [CCP1] (Weinberg, 1989), namely, why is the energy scale of the effective gravitating vacuum energy density ρ_V negligible compared to the basic energy scales of the standard model of elementary particle physics (not to mention the Planck energy scale). A first step towards solving CCP1 has been made using q -theory (Klinkhamer & Volovik, 2010). The third condition is to have a stationary cosmological solution (de-Sitter universe) which is analytic and consistent, as discussed in Section III of Klinkhamer & Volovik (2008b). The second and fourth conditions are for simplicity's

sake, without the need to introduce new physics at ultralow energies [that is, energies of the order of meV, as indicated by the observed accelerating universe; cf. [Riess et al. \(1998\)](#); [Perlmutter et al. \(1999\)](#); [Komatsu et al. \(2009\)](#)].²

Remark that the Λ CDM model [i.e., the cosmological model from standard general relativity with a genuine cosmological constant Λ and a cold-dark-matter (CDM) component] already does not satisfy the first condition. Moreover, the required value of Λ in the Λ CDM model ([Weinberg, 1972](#); [Carroll, Press, & Turner, 1992](#); [Sahni & Starobinsky, 2000](#); [Peebles & Ratra, 2003](#); [Perivolaropoulos, 2010](#)) is completely *ad hoc* (the meV scale mentioned above) and leaves CCP1 hanging in the wind, which is theoretically unsatisfactory, as explained in, e.g., the second paragraph of Section I of [Mukohyama \(2004\)](#). Just to avoid any misunderstanding, the flat Λ CDM model ([Weinberg, 1972](#); [Carroll, Press, & Turner, 1992](#); [Sahni & Starobinsky, 2000](#); [Peebles & Ratra, 2003](#); [Perivolaropoulos, 2010](#)) is perfectly satisfactory experimentally but not theoretically. The QCD-scale modified-gravity model proposed in the next subsection has the potential to be incorporated in a theoretically satisfactory framework (q -theory) but, first, needs to be shown experimentally satisfactory.

2.2. Representative model

The following simplified modified-gravity action ([Klinkhamer & Volovik, 2009a](#); [Klinkhamer, 2010a](#)) satisfies the criteria of Section 2.1 and will be the starting point of the present article:

$$S_{\text{grav},0} = \int_{\mathbb{R}^4} d^4x \sqrt{-g(x)} \left[K_0 R(x) - \eta |R(x)|^{1/2} (q_0)^{3/4} + \mathcal{L}_M[\psi(x)] \right], \quad (2.1)$$

with $\hbar = c = 1$ from the use of natural units, the gravitational coupling constant $K_0 \equiv (16\pi G_0)^{-1} > 0$, the dimensionless coupling constant $\eta > 0$

²Of all four conditions, the third has admittedly the weakest theoretical motivation. But additional arguments may perhaps come from future results on the quantum-field-theoretic de-Sitter state, which may hold some surprises in store [see, e.g., [Polyakov \(2010\)](#) and references therein].

[standard general relativity has $\eta = 0$], the constant equilibrium gluon condensate $q_0 \equiv (E_{\text{QCD}})^4$, and the generalized matter field $\psi(x)$ which includes the fields of QCD (gluons and quarks) and further possible fields responsible for the observed CDM component of the present universe.³ Throughout, the conventions of [Weinberg \(1972\)](#) are used such as the metric signature $(-+++)$, except for the Riemann tensor, which is taken to have a further minus sign. Specifically, the Ricci curvature scalar of a de-Sitter universe is given by $R = 12 H^2 \geq 0$ with Hubble constant H .

It remains to be seen if there exist other $f(R)$ modified-gravity models which satisfy the criteria of [Section 2.1](#) and describe the observed universe equally successfully. Perhaps a model exists involving the electroweak scale ([ArkaniHamed et al., 2000](#); [Klinkhamer & Volovik, 2009b](#); [Klinkhamer, 2010b, 2011](#)). Returning to the energy scale of QCD, remark that the second term of the integrand in [\(2.1\)](#) corresponds to the linear term $(E_{\text{QCD}})^3 H$ mentioned in the second criterium of [Section 2.1](#) and that this linear term may be considered to be the leading term of a power series in H (higher-order terms will be discussed shortly). Anyway, model [\(2.1\)](#) based on the QCD energy scale is a perfect example (essentially unique up to higher-order terms) of a model satisfying the criteria of [Section 2.1](#). As explained in [Section 1](#), the aim of the present article is to study the use of this model as a compact description of the observed universe.

In the standard formulation of modified-gravity models where the Ricci scalar R of the Einstein–Hilbert action density is replaced by $R + f(R)$, the proposed modification is given by

$$f(R) = -|R|^{1/2}/L_0, \quad (2.2a)$$

with length scale

$$L_0 \equiv \eta^{-1} K_0 (q_0)^{-3/4}. \quad (2.2b)$$

It is important to state, right from the start, that the theory [\(2.1\)](#) is only considered to be relevant over cosmological length scales (small curvatures

³As discussed in [Klinkhamer & Volovik \(2009a\)](#), the vacuum energy density of the QCD fields with dynamics governed by \mathcal{L}_M in [\(2.1\)](#) is compensated by an appropriate value q_0 of the gluon-condensate $q(x)$ field, resulting in a vanishing effective cosmological constant. Other (non-QCD) contributions to the gravitating vacuum energy density are perhaps canceled by the self-adjustment of similar q -type fields or by a different mechanism altogether; cf. Ftn. 2 of [Klinkhamer & Volovik \(2009a\)](#).

$|R|$). For smaller length scales (larger curvatures $|R|$), other terms than the single root term (2.2) may become important. One example would be having $f(R)$ of (2.2) replaced by the following extended function:

$$\tilde{f}_{\text{ext}}(R) = -\frac{|R|^{1/2}/L_0}{1 + \zeta |R|^{1/2} L_0}, \quad (2.3)$$

with another positive coupling constant ζ of order unity. This particular function differs from the one given by Klinkhamer (2010a) in that its second derivative is positive for all finite values of R .

2.3. Additional remarks

The model of the previous subsection with a constant value q_0 of the gluon condensate is a simplified version of the model with a dynamic gluon condensate field $q(x)$ considered in Klinkhamer (2010a). It was found in (Klinkhamer, 2010a) that the gravitating vacuum energy density $\rho_V(q) \propto (q - q_0)^2$ of the corresponding cosmological model is negligible during the late evolution of the model universe and that the dynamic vacuum variable $q(x)$ is effectively frozen to its constant equilibrium value q_0 with $\rho_V(q_0) = 0$. For this reason, it makes sense to restrict the consideration to model (2.1) with just a constant value q_0 .

The actual value of the gluon-condensate q_0 entering the nonanalytic modified-gravity term (2.2) arises from the flat-spacetime part of the Lagrange density, \mathcal{L}_M , and is taken to have a standard (positive) value $q_0 \approx (300 \text{ MeV})^4$; see Shifman, Vainshtein & Zakharov (1979); Narison (1996). Later, it will be shown that this particular value for q_0 implies $\eta \approx 2 \times 10^{-4}$. Observe that, strictly speaking, the quantity q_0 of the second term in the integrand of (2.1) need not be equal to the equilibrium value of the gluon condensate $q(x)$, but can, in principle, be given by any typical QCD energy density. Still, the identification of q_0 as the equilibrium gluon condensate is maintained throughout this article, because, with this identification, the coupling constant η in (2.1) is defined unambiguously.

The relation between the gravitational constant $G_0 \equiv G(q_0) \equiv (16\pi K_0)^{-1}$ from (2.1) and Newton's constant G_N (Cavendish, 1798; Mohr, Taylor & Newell, 2008) is rather subtle, but, in this article, the approximate equality $G_0 \sim G_N$ is simply taken to hold (see the next paragraph for the argument). For these numerical values of the constants $\eta (q_0)^{3/4}$ and G_0 , the length scale

entering the modified-gravity action term (2.2) has the value $L_0 \sim 10^{26} \text{ m} \sim 3 \text{ Gpc}$.

The field equations from (2.1) are fourth order and it is advantageous to switch to the scalar-tensor formulation, which has field equations of second order but involves an extra scalar field (Brans & Dicke, 1961; Starobinsky, 1980; Sotiriou & Faraoni, 2010).⁴ The scalar-tensor theory equivalent to (2.1) has been given by Eq. (2.2) of Klinkhamer (2010a), with q replaced by q_0 and changing the sign in front of the ϕR term, where $\phi \in (-\infty, 1)$ is the dimensionless Brans–Dicke scalar field. The corresponding potential $U(\phi) \propto \eta^2/(1 - \phi)$ is such that it leads to the chameleon effect (Khoury & Weltman, 2004) and the effect is taken to be operative for Cavendish-type experiments on Earth (Cavendish, 1798; Mohr, Taylor & Newell, 2008), giving $G_0 \sim G_N$ (see also Endnote [39] in Klinkhamer (2010a) for further discussion).

3. One-component QCD-scale modified-gravity universe

3.1. Pressureless matter component

In order to study cosmological aspects of the modified-gravity model (2.1), consider a spatially flat ($k = 0$) Robertson–Walker metric (Weinberg, 1972) with scale factor $a(\tau)$ for cosmic time τ (later, t will denote the dimensionless quantity) and matter described by homogeneous perfect fluids. Recall that the Hubble parameter is given by $H(\tau) \equiv [da(\tau)/d\tau]/a(\tau)$.

In fact, only one matter component (labeled $n = 2$) will be considered in the rest of this article, namely, a perfect fluid of pressureless nonrelativistic matter [e.g., cold dark matter] with energy density $\rho_{M2}(\tau)$ and constant equation-of-state (EOS) parameter $w_{M2} \equiv P_{M2}/\rho_{M2} = 0$. This is a simplification of the model considered by Klinkhamer (2010a), which had also a dynamic gluon-condensate component (labeled $n = 0$) with EOS parameter $w_0 = -1$ and an ultrarelativistic-matter component (labeled $n = 1$) with

⁴ The modified Einstein equation [see, e.g., Eq. (6) of Sotiriou & Faraoni (2010)] also has singular terms proportional to $|R|^{-1/2}$ or higher negative powers of $|R|$. It remains to be seen whether or not this leads to unacceptable behavior. As mentioned in Klinkhamer (2010a), appropriate versions of the model may satisfy solar-system tests because of the chameleon effect (to be discussed shortly). See also the related discussion in the penultimate paragraph of Section 5.

EOS parameter $w_{M1} = 1/3$. In order to allow for an easy comparison with the results of [Klinkhamer \(2010a\)](#), the label $n = 2$ will be kept for the single matter component considered in this article.

Apart from this cold-dark-matter component, the only other input of the cosmological model is the modified-gravity term from [\(2.1\)](#). Using the scalar-tensor formalism, there is then the auxiliary Brans–Dicke scalar $\phi(\tau)$ to consider, without direct kinetic term but with the nontrivial potential $U(\phi)$ already mentioned in [Section 2.3](#).

3.2. Dimensionless variables and ODEs

The following dimensionless variables t , h , f , u , s , and r can be introduced:

$$\tau \equiv t K_0 / (\eta(q_0)^{3/4}), \quad (3.1a)$$

$$H(\tau) \equiv h(t) \eta(q_0)^{3/4} / K_0, \quad (3.1b)$$

$$U(\tau) \equiv u(t) \eta^2(q_0)^{3/2} / K_0^2, \quad (3.1c)$$

$$\phi(\tau) \equiv s(t), \quad (3.1d)$$

$$\rho_{M2}(\tau) \equiv r_{M2}(t) \eta^2(q_0)^{3/2} / K_0, \quad (3.1e)$$

where, different from [Klinkhamer \(2010a\)](#), the rescaling is done with the combination $\eta(q_0)^{3/4} / K_0 = \eta(q_0)^{3/4} (16\pi G_0)$ appearing in the modified-gravity term [\(2.2a\)](#). All dimensionless quantities are denoted by lower-case Latin letters.

From the action [\(2.1\)](#) in the scalar-tensor formulation ([Klinkhamer, 2010a](#)), there is the following closed system of 4 first-order ordinary differential equations (ODEs) for the 4 dimensionless variables $h(t)$, $s(t)$, $v(t)$, and $r_{M2}(t)$:

$$\dot{h} = -2h^2 - \frac{1}{6} \frac{\partial u}{\partial s}, \quad (3.2a)$$

$$\dot{s} = v, \quad (3.2b)$$

$$\dot{v} = \frac{1}{6} r_{M2} - 3hv - \frac{2}{3} u + \frac{1}{3} s \frac{\partial u}{\partial s}, \quad (3.2c)$$

$$\dot{r}_{M2} = -3hr_{M2}, \quad (3.2d)$$

where the overdot stands for differentiation with respect to the dimensionless

cosmic time t and the dimensionless Brans–Dicke potential u is given by

$$u(t) = -\frac{1}{4} \frac{1}{1-s(t)}. \quad (3.3)$$

With the solution of the above ODEs for appropriate boundary conditions, it is possible to verify *a posteriori* the Friedmann-type equation,

$$h^2 s + h v = (r_{M2} - u)/6, \quad (3.4)$$

which, in general, is guaranteed to hold by the contracted Bianchi identities and energy-momentum conservation (Weinberg, 1972). Hence, if the solution of ODEs (3.2) satisfies (3.4) at one particular time, then (3.4) is satisfied at all times.

The boundary conditions for ODEs (3.2) are obtained by setting $t = t_{\text{start}}$ in the following functions:

$$h^{\text{approx}}(t) = \frac{2}{3} \frac{1}{t} \left(1 + \frac{3\sqrt{3}}{16} t - \frac{405}{512} t^2 \right), \quad (3.5a)$$

$$s^{\text{approx}}(t) = \left(1 - \frac{\sqrt{3}}{4} t + \frac{9}{16} t^2 \right), \quad (3.5b)$$

$$v^{\text{approx}}(t) = \dot{s}^{\text{approx}}(t), \quad (3.5c)$$

$$r_{M2}^{\text{approx}}(t) = \frac{8}{3} \frac{1}{t^2} \left(1 - \frac{3\sqrt{3}}{8} t \right). \quad (3.5d)$$

These functions provide, in fact, an approximate solution of the ODEs (3.2) and generalized Friedmann equation (3.4) in the limit $t \rightarrow 0$.

Purely mathematically, remark that the ODEs (3.2) have no free parameters and that the boundary conditions are fixed completely by (3.5) for sufficiently small t_{start} . Physically, however, t_{start} must be small enough but still larger than the time t_{cross} corresponding to the QCD crossover at a temperature $T_{\text{cross}} \sim E_{\text{QCD}} \sim 300$ MeV [see Klinkhamer (2010a) for further discussion]. Specifically, one must take a t_{start} value obeying $1 \gg t_{\text{start}} \gg t_{\text{cross}} \sim \eta E_{\text{QCD}}/E_{\text{Planck}} \sim 10^{-23}$ for $\eta \sim 10^{-4}$.

3.3. Observables quantities

In order to test the cosmology resulting from QCD-scale modified-gravity model (2.1), four observable quantities will be considered (Klinkhamer, 2010a; Zhang et al., 2007): $t_p h(t_p)$, $\bar{w}_X(t_p)$, $z_{\text{infect}}(t_i, t_p)$, and $E_G^{\text{theo}}(z)$, where z stands for the redshift with respect to the present epoch $t = t_p$ to be defined below. As a preliminary, note that the generalized Friedmann equation (3.4) gives

$$\bar{w}_X + \bar{w}_{M2} = 1, \quad (3.6a)$$

$$\bar{w}_X \equiv r_X/(6h^2) \equiv \bar{w}_V + \bar{w}_{\text{grav}}, \quad (3.6b)$$

$$\bar{w}_V \equiv r_V/(6h^2) = 0, \quad (3.6c)$$

$$\bar{w}_{\text{grav}} \equiv 1 - s - \dot{s}/h - u/(6h^2), \quad (3.6d)$$

$$\bar{w}_{M2} \equiv r_{M2}/(6h^2). \quad (3.6e)$$

Without genuine vacuum energy density [$r_V(q) \equiv K_0 \eta^{-2}(q_0)^{-3/2} \rho_V(q) = 0$ for $q = q_0$], the only new ingredient in (3.6) is \bar{w}_{grav} , as it vanishes for the standard theory with $u = 0$ and $s = 1$. Remark that r_X of (3.6b) does not correspond to a real physical energy density but is a mathematical quantity inferred from the variables \bar{w}_V and \bar{w}_{grav} defined by (3.6c) and (3.6d).

The energy density parameters of (3.6) are defined in terms of the gravitational constant $G_0 \equiv G(q_0)$, which may, in principle, be different from Newton's constant G_N as measured by laboratory experiments on Earth (Cavendish, 1798; Mohr, Taylor & Newell, 2008). For this reason, the parameters have been denoted by ω and not by the standard symbol Ω . Still, the precise relation between G_0 and G_N will be important only once in Section 5, namely, when the absolute age of the universe is calculated and, then, the equality $G_0 = G_N$ is taken to hold [assuming the chameleon effect (Khoury & Weltman, 2004) to be relevant for earth-based experiments, as discussed in Section 2.3]. Now, turn to the four observables mentioned above.

The first observable is simply the expansion rate of the universe in units of the inverse of its age,

$$t_p h(t_p) \equiv t \dot{a}(t)/a(t) \big|_{t=t_p}, \quad (3.7)$$

where $a(t)$ is the scale factor of the Robertson–Walker metric. This observable quantity has been evaluated for the “present epoch,” which is taken to

be defined by the moment $t = t_p$ when $\bar{\omega}_{M2}(t_p) = 1/4$ or $\rho_X(\tau_p)/\rho_{M2}(\tau_p) = 3$ from (3.6a). The fiducial value $\bar{\omega}_{M2}(t_p) = 1/4$ will be used in the following and is more or less consistent with the available data [see, e.g., Komatsu et al. (2009)].

The second observable corresponds to the effective EOS parameter of the unknown component X , whose model value can be extracted from (3.2) and (3.4):

$$\begin{aligned}\bar{w}_X(t_p) &\equiv -\frac{2}{3} \left(\frac{\ddot{a} a}{(\dot{a})^2} + \frac{1}{2} \right) \frac{1}{1 - \bar{\omega}_{M2}} \Big|_{t=t_p} \\ &= -\frac{u + 4h\dot{s} + 2\ddot{s}}{u + 6h\dot{s} - r_{M2}(1-s)} \Big|_{t=t_p}.\end{aligned}\quad (3.8)$$

Again, this observable quantity has been evaluated at the present epoch $t = t_p$. For later times, the right-hand side of (3.8) shows that \bar{w}_X of the modified-gravity model (2.1) approaches the value -1 in the limit of vanishing matter content r_{M2} and constant Brans–Dicke scalar s as $t \rightarrow \infty$.

The third type of observable follows from the transition of deceleration to acceleration. In mathematical terms, this time corresponds to the nonstationary inflection point of the function $a(t)$, that is, the value t_i at which the second derivative of $a(t)$ vanishes but not the first derivative. Specifically, the inflection point $t = t_i$ corresponds to the following redshift for an observer at $t = t_p \geq t_i$:

$$z_{\text{inflect}}(t_i, t_p) \equiv a(t_p)/a(t_i) - 1. \quad (3.9)$$

In order to prepare for the fourth observable of the empirical model (2.1), turn to the linear growth of sub-horizon matter-density perturbations in the Newtonian gauge (Song, Hu & Sawicki, 2007). For small enough matter-density-perturbation amplitudes $\Delta_{M2} \equiv \delta r_{M2}/r_{M2}$ and large enough wavelengths (but still within the horizon), the following dimensionless linear ODE needs to be solved:

$$\ddot{\Delta}_{M2}(t) + 2h(t)\dot{\Delta}_{M2}(t) - \frac{1}{3} \frac{1}{s(t)} r_{M2}(t) \Delta_{M2}(t) = 0, \quad (3.10)$$

which is equivalent to, for example, Eq. (13) of Tsujikawa et al. (2009) or Eq. (15) of Lombriser et al. (2010), with an extra factor $(4/3)(1/\phi)$ multiplying the original non-derivative term. See also de la Cruz-Dombriz, Dobado

& Maroto (2008), where the approximate ODE (3.10) has been tested for an $|R|^{1/2}$ modified-gravity model.

From the combined solutions for $a(t)$ and $\Delta_{M2}(t)$, the following linear growth parameter is obtained by differentiation:

$$\beta \equiv \frac{d \ln \Delta_{M2}}{d \ln a} = \frac{1}{h} \frac{\dot{\Delta}_{M2}}{\Delta_{M2}}. \quad (3.11)$$

For the early phase with $\phi = s \sim 1$, $h \sim 2/3 t^{-1}$, and $r_{M2} \sim 6 h^2$, the linear growth parameter calculated from (3.10) is

$$\beta_{\text{early}} = (\sqrt{33} - 1)/4 \approx 1.186. \quad (3.12)$$

The corresponding matter-density perturbation is given by

$$\Delta_{M2}^{\text{approx}}(t) \propto t^{2\beta_{\text{early}}/3}, \quad (3.13)$$

which provides an approximate solution of the ODE (3.10) in the limit $t \rightarrow 0$.

Now, consider the estimator E_G introduced by Zhang et al. (2007) with the purpose of searching for deviations from Einstein gravity. According to the original paper of Zhang et al. (2007) and also the paper of Lombriser et al. (2010), the theoretical expression for $E_G(z)$ in the modified-gravity model (2.1) is

$$E_G^{\text{theo}}(z) = \frac{\bar{\omega}_{M2}(t_p)}{(1 + f_R(z)) \beta(z)} = \frac{\bar{\omega}_{M2}(t_p)}{\phi(z) \beta(z)}, \quad (3.14)$$

with $f_R \equiv df/dR$. The last expression of (3.14) is given in terms of the scalar-tensor formalism, where the relevant field equations (Klinkhamer, 2010a) have been used. From the numerical solution of the ODEs to be presented in Section 5, the values of $\phi(z)$ and $\beta(z)$ are readily obtained and, thereby, the numerical value of $E_G^{\text{theo}}(z)$. In the early phase with $\phi \sim 1$ and β given by (3.12), one has

$$E_G^{\text{theo}}|_{z \gg 1} \sim \bar{\omega}_{M2}(t_p)/\beta_{\text{early}} \approx 0.2108, \quad (3.15)$$

for the fiducial value $\bar{\omega}_{M2}(t_p) = 0.25$ mentioned earlier.

4. Benchmark results from the Λ CDM model

Four observables of the QCD-scale modified-gravity model (2.1) have been discussed in Section 3.3 and will be evaluated numerically in Section 5. In

this section, the corresponding values are given for the spatially flat Λ CDM-model universe (Weinberg, 1972; Carroll, Press, & Turner, 1992; Sahni & Starobinsky, 2000; Peebles & Ratra, 2003; Perivolaropoulos, 2010).

Recall that the relevant spatially flat Λ CDM-model is completely defined by the value of the cosmological constant Λ and the condition that Ω_{M2} equals 1/4 at present. Here, the standard matter energy-density parameter in terms of Newton's constant G_N is given by Weinberg (1972)

$$\Omega_{M2}(\tau_p) \equiv \rho_{M2}(\tau_p)/\rho_{\text{crit}}(\tau_p), \quad (4.1a)$$

$$\rho_{\text{crit}}(\tau_p) \equiv 3H^2(\tau_p)/(8\pi G_N), \quad (4.1b)$$

where the present epoch occurs at cosmic time $\tau = \tau_p$. As mentioned in Section 2.1, the Λ CDM model may be theoretically unsatisfactory but has been found to give an excellent description of the observed accelerating universe (Riess et al., 1998; Perlmutter et al., 1999; Komatsu et al., 2009). For this reason, it can provide benchmark results to compare other models with.

Analytic results for the first three observables of Section 3.3 are as follows (Klinkhamer, 2010a):

$$\begin{aligned} & \left(\tau_p H(\tau_p), \overline{w}_X(\tau_p), z_{\text{inflect}}(\tau_i, \tau_p) \right) \Big|_{\Lambda\text{CDM}} \\ &= \left((4 \operatorname{arcsinh} \sqrt{3})/(3 \sqrt{3}), -1, (6^{1/3} - 1) \right) \\ &\approx (1.01, -1, 0.817). \end{aligned} \quad (4.2)$$

The fourth observable is simply (Zhang et al., 2007)

$$E_G^{\text{theo}}(z) \Big|_{\Lambda\text{CDM}} = \Omega_{M2}(\tau_p)/\beta_{\Lambda\text{CDM}}(z), \quad (4.3)$$

where $\beta_{\Lambda\text{CDM}}(z)$ can be calculated analytically from the exact solution of the linear ODE corresponding to (3.10); see, in particular, the C_2 term in Eq. (6.67) of Mukhanov (2005). The estimator (4.3) approaches the constant value $\Omega_{M2}(\tau_p)$ for increasing redshift z , as $\beta_{\Lambda\text{CDM}}(z)$ goes to 1 for $z \rightarrow \infty$.

5. Numerical results for the modified-gravity universe

Figure 1 displays the numerical solution of the first-order nonlinear ODEs (3.2) with boundary conditions (3.5) at $t = t_{\text{start}} = 10^{-5}$. The simultaneous

numerical solution of the second-order linear ODE (3.10) has been obtained for boundary conditions from the approximate solution (3.13) at $t = t_{\text{start}}$. Specifically, (3.13) provides the initial derivative $\dot{\Delta}_{M2}(t_{\text{start}})$ for a given initial value of $\Delta_{M2}(t_{\text{start}})$. Numerical solutions of (3.10) with other values for the initial derivative $\dot{\Delta}_{M2}(t_{\text{start}})$ have been seen to rapidly approach the approximate solution (3.13), provided the dimensionless cosmic time t remains small enough.

The linear t scale of Fig. 1 is convenient for the late evolution of the model universe, because, as will be seen shortly, t corresponds to the cosmic time τ measured in units of approximately $10^{10} \text{ yr} = 10 \text{ Gyr}$. A logarithmic scale is, however, more appropriate for the early phase and Fig. 2 shows that the numerical results from the QCD-scale modified-gravity model (2.1) reproduce the Friedmann–Robertson–Walker–type expansion with Hubble parameter $h = 2/3 t^{-1}$ and linear growth parameter β given by (3.12). Table 1 displays, moreover, a stable behavior of the $t = t_p$ observables within the numerical accuracy (at the one-per-mill level or better, for the quantities shown).

The results for the first three observable quantities of Section 3.3 are as follows:

$$\begin{aligned} & \left(\tau_p H(\tau_p), \bar{w}_X(\tau_p), z_{\text{inflect}}(\tau_i, \tau_p) \right) \\ & \approx (0.917, -0.662, 0.523). \end{aligned} \quad (5.1)$$

These results may be compared to the Λ CDM–model values (4.2).

The obtained values (5.1) are consistent with those of Table I in Klinkhamer (2010a) for the modified-gravity model with a dynamic q field, but without the need to consider the limit of the mathematical parameter $Z \equiv (q_0)^{1/2} K_0^{-1} \equiv 2 (E_{\text{QCD}}/E_{\text{Planck}})^2$ to a numerical value of order 10^{-38} , as Z has been scaled away completely in the present simplified model. The only place where this hierarchy parameter Z enters is for the dimensional age of the present universe and its expansion rate.

In fact, with $G_0 = G_N$ and the elementary-particle-physics result (Shifman, Vainshtein & Zakharov, 1979; Narison, 1996) for the flat-spacetime gluon condensate $q_0 \equiv (E_{\text{QCD}})^4 = (300 \text{ MeV})^4$, the values $(t_p, h_p) = (1.374, 0.6673)$ from the numerical solution (Table 1) give the following results for

the present age and expansion rate of the model universe:

$$\begin{aligned}
\tau_p &= t_p \eta^{-1} K_0 (q_0)^{-3/4} \\
&= t_p \eta^{-1} (16\pi G_N)^{-1} (E_{\text{QCD}})^{-3} \\
&\approx 13.2 \text{ Gyr},
\end{aligned} \tag{5.2a}$$

$$\begin{aligned}
H_p &= h_p \eta K_0^{-1} (q_0)^{3/4} \\
&= h_p \eta (16\pi G_N) (E_{\text{QCD}})^3 \\
&\approx 68.1 \text{ km s}^{-1} \text{ Mpc}^{-1},
\end{aligned} \tag{5.2b}$$

$$\eta \approx 2.40 \times 10^{-4}, \tag{5.2c}$$

where the numerical values given in (5.2a) and (5.2b) use the η value given in (5.2c). The modified-gravity model (2.1) can thus give reasonable values for *both* the present age and expansion rate, which traces back to the fact that the numerical value for the product $t_p h_p$ has been found to be close to 1. Only the actual value of η depends on the precise determination of q_0 and G_0/G_N . All other results of this paper do not depend on η , q_0 , or G_0 as these quantities can be scaled out according to (3.1). The rest of this section focusses on two such observable quantities, $H(z)/H(0)$ and $E_G(z)$, both considered as a function of the redshift z .

The behavior of the Hubble parameter $H(z)$ relative to the calculated present value (5.2b) is given by the second column of Table 2. The corresponding modified-gravity results for $H(z)$ agree with the observations reported in Fig. 9 and Table 2 of Stern et al. (2010), even though the scatter of the data is still substantial. The third column of Table 2 gives, for comparison, the Hubble-parameter ratios from the Λ CDM model, which also agree with the current $H(z)$ observations (Stern et al., 2010) if the measured value of $H(0)$ is used as input.

The numerical solution of Fig. 1 at $t = 1.0011$ gives the redshift $z \approx 0.32$, the scalar-field value $\phi = s \approx 0.7666$, and the linear growth parameter $\beta \approx 0.7459$. These numbers combined with the fiducial value $\bar{\omega}_{M2p} = 0.25$ result in the following numerical value for the gravity estimator (3.14):

$$E_G^{\text{theo}} \big|_{z=0.32} \approx 0.437, \tag{5.3}$$

which agrees within one sigma with the experimental result $E_G^{\text{exp}} = 0.392 \pm 0.065$ from Reyes et al. (2010) for a sample of galaxies with an average redshift $\langle z \rangle = 0.32$.

The same quantity E_G^{theo} at $z \approx 0.3$ has also been calculated by [Lombriser et al. \(2010\)](#) for an $f(R)$ modified-gravity model with $1 + f_R = \phi \sim 1$ and B_0 parameter ([Song, Hu & Sawicki, 2007](#)) of order 0.2, giving a value around 0.35. In order to compare to this result, the value of the B_p parameter (in the notation with the present epoch at $\tau = \tau_p$) has been calculated for the QCD-scale modified-gravity model (2.1):

$$\begin{aligned} B_p &\equiv \frac{R f_{RR}}{1 + f_R} \left(\frac{1}{R} \frac{dR}{d \ln a} \right) \left(H \frac{d \ln a}{dH} \right) \Big|_{\tau=\tau_p} \\ &= \frac{1}{2} \frac{1}{2 L_0 \sqrt{|R|} - 1} \left(\frac{\dot{R}}{R} \frac{H}{\dot{H}} \right) \Big|_{\tau=\tau_p} \\ &\approx 0.246, \end{aligned} \tag{5.4}$$

where the Ricci scalar is given by $R = 6(dH/d\tau + 2H^2)$ and the function $f(R)$ is defined by (2.2). With this B_p parameter and simply omitting the factor $1/(1 + f_R)$ or $1/\phi$ in (3.14), our result for E_G at $z = 0.32$ would be approximately 0.34, which would agree with the result of Fig. 4 in [Lombriser et al. \(2010\)](#).

Nevertheless, the factor $1/(1 + f_R)$ is unarguably present in the E_G expression (3.14) and the correct estimate from the QCD-modified-gravity model is (5.3), which is larger than that of [Lombriser et al. \(2010\)](#) but still consistent with the direct measurement ([Reyes et al., 2010](#)). Observe that the value $f_R(t_p) \approx 0.7259 - 1 \approx -0.2741$ is consistent with the cosmic-microwave-background data according to Table III of [Lombriser et al. \(2010\)](#), but apparently inconsistent with the cluster-abundance data according to Table IV of the same reference. However, as mentioned before, the QCD-modified-gravity model is assumed to hold only for the very largest length scales and not galaxy-cluster length scales (or, *a fortiori*, solar-system length scales). Interestingly, the cluster-abundance data can be used to constrain the extensions of the simple $|R|^{1/2}$ term, one possibility having been mentioned in (2.3).

Finally, additional values for E_G^{theo} at selected redshifts are given in the fourth column of Table 2. This table also compares the QCD-modified-gravity results for E_G^{theo} with those of the Λ CDM-model universe. [The Λ CDM value for E_G^{theo} at $z = 0.32$ is 0.396, whereas the modified-gravity value has already been given in (5.3) and the experimental value just below

that equation.] Putting the theoretical predictions for $E_G(z)$ in Table 2 next to the simulated data in Fig. 1 of Zhang et al. (2007) suggests that it may be difficult for future surveys (e.g., the Square Kilometer Array on the ground or the Joint Dark Energy Mission and the Euclid Satellite in space) to distinguish between the two theoretical models but perhaps not impossible.

6. Discussion

In this article, a simple empirical model has been proposed with a QCD-scale modified-gravity term (2.1) and a single pressureless matter component (cold dark matter, CDM). This particular $f(R)$ modified-gravity model has been selected on physical grounds (Section 2.1), but is, in the end, solely used as an efficient way to describe the main aspects of the late evolution of the universe, having only two fundamental energy scales, E_{QCD} and E_{Planck} , and a single dimensionless coupling constant, η .

With the elementary-particle-physics value for the equilibrium gluon condensate $q_0 \equiv (E_{\text{QCD}})^4 = (300 \text{ MeV})^4$, the measured age of the universe fixes the dimensionless coupling constant η of model (2.1) to the value of approximately 2.4×10^{-4} . As emphasized in Klinkhamer (2010a), the effective coupling constant η may ultimately be calculated from QCD and general relativity; see also Schützhold (2002); Bjorken (2004); Urban & Zhitnitsky (2010a,b); Holdom (2011) for further discussion of the possible relation of QCD and dark energy. The connection of the QCD-scale modified-gravity model with the q -theory approach to solving the main cosmological constant problem has already been mentioned in Section 2.1.

The q_0 and η values quoted in the previous paragraph result in a model prediction for the Hubble constant (5.2b), which is within 10 % of the observed value. Theoretically, the last mathematical expression for the Hubble constant in the middle of (5.2b) is quite remarkable, as it involves only Newton's constant G_N and the cube of the energy scale E_{QCD} . Independent of the q_0 and η values, there are further model predictions for the dimensionless quantities (5.1) and (5.3), which are again in the same ball park as the observed values. The modified-gravity model prediction of -0.7 for the quantity $\overline{w}_X(\tau_p)$ as defined by (3.8) [compared to the Λ CDM value of -1] may perhaps provide a crucial test, as long as *independent* measurements of the present values of H , $dH/d\tau$, and $\rho_{\text{CDM}} \equiv \rho_{\text{M2}}$ can be obtained.

Moreover, model predictions for the redshift dependence of the gravity

estimator E_G have been given in Table 2. At the very largest length scales (comoving wavelengths of the order of hundred Mpc or more but still less than the horizon scale) and relatively low redshifts $z \sim 0.75$, the QCD-modified-gravity results for E_G differ by some +10% from the Λ CDM-model values. The QCD-modified-gravity results for E_G differ by some -10% from the Λ CDM-model values for $z \gg 3$, but it is not yet clear how these redshifts can be probed observationally.

As it stands, the QCD-scale modified-gravity model (2.1) gives a remarkable description of the main aspects of the late evolution of the universe. With many forthcoming ground-based experiments and space missions, future measurements of $\overline{w}_X(\tau_p)$ and $E_G(z)$ may suggest alternative $f(R)$ functions, such as the one given by (2.3). Of course, if experiment indeed finds evidence for nonzero $f(R)$, it is up to theory to provide the proper understanding.

Acknowledgments

The author thanks S. Appleby and J. Weller for pointing out a problem with an earlier version of Eq. (2.3) and both ASR referees for constructive comments. He also gratefully acknowledges the hospitality of the IPMU during the month of May 2010. This work was supported in part by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

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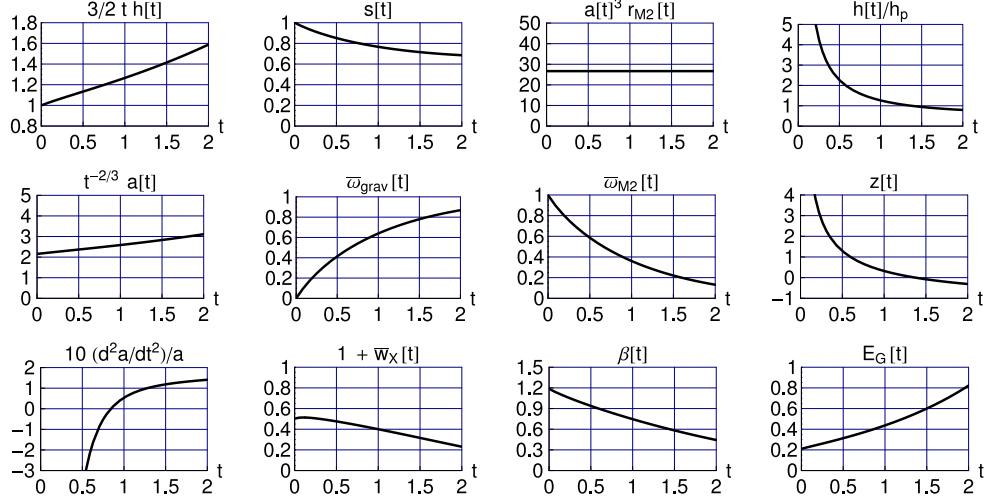


Figure 1: Numerical solution of the modified-gravity cosmological ODEs (3.2), with Brans–Dicke scalar potential (3.3) and pressureless matter having a dimensionless energy density r_{M2} [see (3.1) for the definitions of the dimensionless variables used]. The boundary conditions follow from the approximate solution (3.5) evaluated at $t_{\text{start}} = 10^{-5}$. The initial value $a(t_{\text{start}})$ for the scale factor $a(t)$ is taken as 10^{-3} but has no direct physical relevance. The linear matter-density-perturbation ODE (3.10) is solved simultaneously with boundary conditions from (3.13) evaluated at t_{start} . The figure panels are organized as follows: the panels of the first column from the left concern the scale factor $a(t)$ and the Hubble parameter $h \equiv (da/dt)/a$, those of the second column the Brans–Dicke scalar $s(t)$ [the QCD-modified-gravity model (2.1) being studied in the scalar-tensor formalism], those of the third column the matter energy density r_{M2} [the bottom panel of this column showing the linear growth parameter β of subhorizon matter-density perturbations], and those of the fourth column derived quantities [the bottom panel showing the gravity estimator E_G^{theo} defined by (3.14)]. The three panels of the fourth column can be combined to give the behavior of $h(z)/h(0)$ and $E_G(z)$ shown in Table 2. The several energy-density parameters $\bar{\omega}$ and the effective “dark-energy” equation-of-state parameter $\bar{\omega}_X$ are defined in (3.6) and (3.8), respectively.

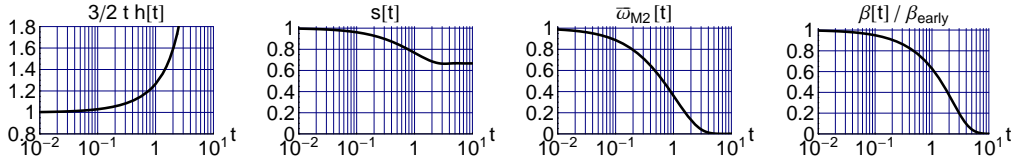


Figure 2: Semi-log plot of numerical quantities from Fig. 1, with β_{early} defined by (3.12).

Table 1: Function values for the “present epoch” [defined by $\bar{\omega}_{M2}(t_p) = 0.25$] from the numerical solution of the modified-gravity cosmological ODEs (3.2), with Brans–Dicke scalar potential (3.3) and boundary conditions (3.5) taken at different values of t_{start} . The values for the linear growth parameter β of subhorizon matter-density perturbations follow from the simultaneous numerical solution of ODE (3.10) with boundary conditions from (3.13) evaluated at t_{start} .

t_{start}	t_p	$\beta(t_p)$	$s(t_p)$	$t_p h(t_p)$	$\bar{w}_X(t_p)$	$z_{\text{inflect}}(t_i, t_p)$
10^{-3}	1.37354	0.621646	0.725876	0.916630	−0.662261	0.522508
10^{-4}	1.37354	0.621646	0.725876	0.916630	−0.662258	0.522527
10^{-5}	1.37354	0.621646	0.725876	0.916630	−0.662283	0.522576
10^{-6}	1.37354	0.621646	0.725876	0.916630	−0.662271	0.522511

Table 2: Numerical modified-gravity results from Fig. 1 for the Hubble parameter $H(z)$ relative to its present value $H(0)$ and the gravity estimator $E_G(z)$ defined by (3.14), taking $\bar{\omega}_{M2}(t_p) = 0.25$ for the $E_G(z)$ values quoted. The E_G value for redshift $z = 10^2$ is essentially equal to the analytic result (3.15). For comparison, also values are given for the spatially flat Λ CDM model with $\Omega_{M2}(\tau_p) = 0.25$ and $E_G(z)$ defined by (4.3).

z	$H(z)/H(0) \big _{\text{mod-grav}}$	$H(z)/H(0) \big _{\Lambda\text{CDM}}$	$E_G^{\text{theo}} \big _{\text{mod-grav}}$	$E_G^{\text{theo}} \big _{\Lambda\text{CDM}}$
0	1.00	1.00	0.554	0.541
0.25	1.20	1.11	0.456	0.418
0.5	1.43	1.26	0.399	0.355
1	1.94	1.66	0.335	0.298
1.5	2.51	2.16	0.301	0.275
2	3.15	2.74	0.281	0.265
2.5	3.83	3.39	0.267	0.259
3	4.57	4.09	0.257	0.256
5	7.94	7.40	0.237	0.252
10	18.9	18.3	0.222	0.250
10^2	508	508	0.211	0.250